Derivatives–Part Two

Given a function $z = f(x,y)$ and a point (x_0,y_0) in its domain, there are infinitely many directional derivatives for f at this point–one for every possible unit vector that we may place at (x_0,y_0). Of all these directional derivatives, *two* have special significance. Their importance is partly due to certain properties and theorems related to them, but they are also significant because these are the two that are *easiest to find*. The two special directional derivatives are the derivative in the direction of $\mathbf{i} = < 1,0 >$ and the derivative in the direction of $\mathbf{j} = < 0,1 >$.

- $D_{\mathbf{i}} f(x_0,y_0)$ is known as the **partial derivative of f with respect to x at (x_0,y_0)**. It is denoted $f_x(x_0,y_0)$ or $\frac{\partial f}{\partial x} \mid_{(x_0,y_0)}$.
- $D_{\mathbf{j}} f(x_0,y_0)$ is known as the **partial derivative of f with respect to y at (x_0,y_0)**. It is denoted $f_y(x_0,y_0)$ or $\frac{\partial f}{\partial y} \mid_{(x_0,y_0)}$.

Here is how we find $f_x(x_0,y_0)$:

1. Start with the formula expressing $f(x,y)$ in terms of the two independent variables x and y.
2. Treat y as if it were a constant, and differentiate the formula with respect to x.
3. Evaluate the resulting formula at the point (x_0,y_0), i.e., substitute x_0 in place of x and y_0 in place of y, then do the math.

Here is how we find $f_y(x_0,y_0)$:

1. Start with the formula expressing $f(x,y)$ in terms of the two independent variables x and y.
2. Treat x as if it were a constant, and differentiate the formula with respect to y.
3. Evaluate the resulting formula at the point (x_0,y_0), i.e., substitute x_0 in place of x and y_0 in place of y, then do the math.

Note that in the case of both partial derivatives, we must *differentiate before we evaluate*. This is exactly the same principle we learned in Calculus I. For instance, say you want to find the slope of the tangent line to the curve $y = x^3$ at the point $(2,8)$. You first differentiate y with respect to x, giving you $y' = 3x^2$. Then you substitute 2 for x, giving you $y' = 3(2)^2 = 12$. You cannot substitute 2 in place of x until *after* you have differentiated!

When finding either partial derivative, the result of Step 2 is a formula which, in general, will involve both x and y. In any particular case, it is possible that either variable could drop out, leaving a formula involving only one variable. It is even possible that both variables will drop out, leaving a constant. However, the general case is a formula involving both x and y.

- When partially differentiating with respect to x, the result of Step 2 is denoted $f_x(x,y)$ or $\frac{\partial f}{\partial x}$. For brevity, we may refer to this as f_x. Since $z = f(x,y)$, we may write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$.

- When partially differentiating with respect to y, the result of Step 2 is denoted $f_y(x,y)$ or $\frac{\partial f}{\partial y}$. For brevity, we may refer to this as f_y. Since $z = f(x,y)$, we may write $\frac{\partial z}{\partial y}$ in place of $\frac{\partial f}{\partial y}$.

For $f(x,y) = x^2 + y^2$:

- $f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = D_x(x^2 + y^2) = 2x$
- $f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = D_y(x^2 + y^2) = 2y$
- At the point $(2,3)$, we get $f_x(2,3) = \frac{\partial f}{\partial x}\ |_{(2,3)} = 2(2) = 4$ and $f_y(2,3) = \frac{\partial f}{\partial y}\ |_{(2,3)} = 2(3) = 6$. These are the same results we found ealier (but with much more difficulty then).

We mentioned ealier that the surface $z = x^2 + y^2$ has a tangent plane at the point $(2,3,13)$, and we showed that the equation of the tangent plane was $4x + 6y - z = 13$. Note that the coefficient of x is $f_x(2,3)$ and the coefficient of y is $f_y(2,3)$. Thus, the left side of the equation can be expressed $f_x(2,3)x + f_y(2,3)y - z$. As we shall see later, this is a general rule: If a function $z = f(x,y)$ has a tangent plane at (x_0,y_0), then the left side of the equation of the tangent plane will be $f_x(x_0,y_0)x + f_y(x_0,y_0)y - z$. What about the right side of the equation? In this case, it is simply z_0. But that is not a general rule. In general, the right side may be more complicated. We shall postpone further discussion of this topic until a later section.

The process of finding a partial derivative is known as **partial differentiation**.

- The process of partially differentiating with respect to x is symbolized by the **partial differentiation operator** $\frac{\partial}{\partial x}$ or D_x.
- The process of partially differentiating with respect to y is symbolized by the **partial differentiation operator** $\frac{\partial}{\partial y}$ or D_y.

For example, we can write $\frac{\partial}{\partial x}(5x^2 - 3xy + 7y^2) = 10x - 3y$, and $D_y(5x^2 - 3xy + 7y^2) = -3x + 14y$.

The prior example illustrates an important point. When partially differentiating with respect to one variable, we treat the other variable as a *constant*, but how we deal with a constant depends on whether it is a constant *factor* or a constant *term*. As we know from Calculus I, when differentiating, a constant factor *factors out*, whereas a constant term *drops out* (i.e., goes to zero).

Let us further examine the function $f(x,y) = x^2 + y^2$, but shift our attention from the point $(2,3)$ to another point, let's say the point $(-7,13)$. Then:

- $f_x(-7,13) = \frac{\partial f}{\partial x}\ |_{(-7,13)} = 2(-7) = -14$
- $f_y(-7,13) = \frac{\partial f}{\partial y}\ |_{(-7,13)} = 2(13) = 26$

Let us now consider a completely fresh example:
$z = f(x,y) = 3x^5 - 7x^2y^4 + 9y^2 + 4x - 6y + 12$.

- $f_x(x,y) = \frac{\partial f}{\partial x} = 15x^4 - 14xy^4 + 4$

- $f_y(x,y) = \frac{\partial f}{\partial y} = -28x^2y^3 + 18y - 6$
- $f_x(6,-2) = \frac{\partial f}{\partial x} \mid_{(6,-2)} = 18,100$
- $f_y(6,-2) = \frac{\partial f}{\partial y} \mid_{(6,-2)} = 8,022$

Partial differentiation may be used in concert with implicit differentiation. See Example 5 on page 917.

The functions $f_x(x,y)$ and $f_y(x,y)$ are known as the **first-order partial derivatives** of f. We can also find the **second-order partial derivatives** of f. To accomplish this, we partially differentiate the first-order partial derivatives. Furthermore, we can find the **third-order partial derivatives** of f. To accomplish this, we partially differentiate the second-order partial derivatives. And so on ad infinitum.

In principle, f has four second-order partial derivatives:

1. $\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y)$

2. $\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy}(x,y)$

3. $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x\,\partial y} = f_{yx}(x,y)$

4. $\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = f_{yy}(x,y)$

For $f(x,y) = x^2 + y^2$, since $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$, we will have $f_{xx}(x,y) = 2$, $f_{xy}(x,y) = 0$, $f_{yx}(x,y) = 0$, and $f_{yy}(x,y) = 2$.

For $f(x,y) = 3x^5 - 7x^2y^4 + 9y^2 + 4x - 6y + 12$, since $f_x(x,y) = 15x^4 - 14xy^4 + 4$ and $f_y(x,y) = -28x^2y^3 + 18y - 6$, we will have $f_{xx}(x,y) = 60x^3 - 14y^4$, $f_{xy}(x,y) = -56xy^3$, $f_{yx}(x,y) = -56xy^3$, and $f_{yy}(x,y) = -84x^2y^2 + 18$.

In both of the above examples, we had $f_{xy}(x,y) = f_{yx}(x,y)$. This is not a coincidence. So long as $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are both continuous in an open region, then $f_{xy}(x,y) = f_{yx}(x,y)$ in that region. This is known as **Clairaut's Theorem**.

The Gradient Vector

Our next goal is to find an efficient means for calculating directional derivatives for unit vectors other than **i** or **j**. To accomplish this, we must introduce a key concept, known as the *gradient vector*.

For any function $f(x,y)$, its **gradient vector** is denoted ∇f, and is defined as $\nabla f = < f_x(x,y),\, f_y(x,y) > = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. If this vector is evaluated at a point (x_0,y_0), we obtain $\nabla f(x_0,y_0) = < f_x(x_0,y_0),\, f_y(x_0,y_0) >$.

For $f(x,y) = x^2 + y^2$, $\nabla f = < 2x, 2y >$, so $\nabla f(2,3) = < 4,6 >$ and $\nabla f(-7,13) = < -14,26 >$.

For the function $f(x,y) = 3x^5 - 7x^2y^4 + 9y^2 + 4x - 6y + 12$,
$\nabla f = < 15x^4 - 14xy^4 + 4, -28x^2y^3 + 18y - 6 >$, so $\nabla f(6,-2) = < 18,100,\ 8,022 >$.

For any unit vector $\mathbf{u}$, if we wish to find $D_{\mathbf{u}} f(x_0,y_0)$, simply compute the dot product of $\mathbf{u}$ and $\nabla f(x_0,y_0)$. In other words, $D_{\mathbf{u}} f(x_0,y_0) = \mathbf{u} \cdot \nabla f(x_0,y_0)$.

For instance, if $f(x,y) = x^2 + y^2$ and $\mathbf{u} = < 0.6, 0.8 >$, then $D_{\mathbf{u}} f(2,3) = < 0.6, 0.8 > \cdot < 4,6 >$ $= 2.4 + 4.8 = 7.2$.

We already had this result. Let's find a directional derivative where we don't already know the answer. If $f(x,y) = x^2 + y^2$ and $\mathbf{u} = < \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} >$, then $D_{\mathbf{u}} f(2,3) = < \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} > \cdot < 4,6 > =$ $\frac{4}{\sqrt{2}} + \frac{-6}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$, or $-\sqrt{2}$.

For the function $f(x,y) = x^2 + y^2$, $\nabla f(-7,13) = < -14, 26 >$. Let us use this to calculate two directional derivatives at $(-7,13)$.

- For $\mathbf{u} = < 0.6, 0.8 >$, $D_{\mathbf{u}} f(-7,13) = < 0.6, 0.8 > \cdot < -14, 26 > = -8.4 + 20.8 = 12.4$
- For $\mathbf{u} = < \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} >$, $D_{\mathbf{u}} f(-7,13) = < \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} > \cdot < -14, 26 > = \frac{-14}{\sqrt{2}} + \frac{-26}{\sqrt{2}} = \frac{-40}{\sqrt{2}}$, or $-20\sqrt{2}$.

For the function $f(x,y) = 3x^5 - 7x^2y^4 + 9y^2 + 4x - 6y + 12$, $\nabla f(6,-2) = < 18,100,\ 8,022 >$. Let us use this to calculate one of its directional derivatives at $(6,-2)$.

- For $\mathbf{u} = < \frac{1}{2}, \frac{\sqrt{3}}{2} >$, $D_{\mathbf{u}} f(6,-2) = < \frac{1}{2}, \frac{\sqrt{3}}{2} > \cdot < 18,100,\ 8,022 > = 9,050 + 4,011\sqrt{3}$